Extra questions on graph plotting

These questions are optional extras to the IA Nat. Sci. maths course. It may help to check your plots by comparing them to those produced by a computer or graphical calculator.

Y. Some Graphs to Plot

Y1. Consider $y = x^2 + 4x$.

a) Find the intersections of the curve with the axis.

b) State what the curve does for large and small x.

c) By considering $\frac{\mathrm{d}y}{\mathrm{d}x}$, work out the co-ordinates of the turning point. State which sort of turning point it must be given the overall shape.

d) Sketch the graph.

e) Confirm that the turning point is the kind decided on above by checking the sign of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$.

f) By completing the square, find α and β such that $y = (x - \alpha)^2 + \beta$. State what transformation of the parabola $y = x^2$ this represents, and identify a feature on your sketch which has undergone the appropriate transformation.

Y2. Consider $y = x^3 - 8x^2 + 19x - 12$.

a) State the y-intercept of the graph of this cubic.

b) Fully factorize the cubic, and hence find its zeros, the points at which it intersects the x-axis.

c) What happens to y as x goes to $+\infty$ and $-\infty$? How do you know?

d) About where (without differentiating) do you expect a maximum and a minimum to be?

e) By differentiating, confirm that the maximum and minimum are in the ranges expected, and find their exact positions. (There's no need to find their y-values.)

f) Find the point of inflexion of the graph.

g) Sketch the graph.

Y3. Consider

$$y = \frac{x-5}{x^2-2x-3} \quad .$$

a) Work out the y-intercept of the graph.

b) Find any values of x at which y is zero, and, by factorizing the denominator, find any values of x at which y is infinite (formally undefined). Note that at each of these points y must change sign.

c) Divide the domain (the range of possible x-values) into four regions, with the division points occurring at the points found in the previous part. For each region, work out whether y is positive or negative.

d) Work out what y does as x tends to $+\infty$ and $-\infty$.

e) Using just the information found so far, draw the asymptotes and provide an approximate sketch of the function. Note carefully whether the function approaches $+\infty$ or $-\infty$ at each side of each asymptote.

f) When drawing the sketch above, you should have been forced to include two turning points. By differentiating y, find the exact positions (x-values) of these turning points, and mark them on the sketch. (By doing this, you are also confirming that there aren't any other turning points that you couldn't have guessed beforehand.)

Y4. Consider

$$y = \frac{x}{e^x - e} \quad .$$

a) Find the y-intercept of the graph. Identify any x-intercepts (zeros) and any vertical asymptotes (infinities) of the function.

b) Find the sign of the function for each of the sub-domains created by dividing the domain at the points found above (as in the previous question).

c) Determine what y does as x approaches $+\infty$.

d) Consider what happens as x approaches $-\infty$. Which part of the function is tending to zero? Which straight line forms the asymptote? Thinking carefully about the signs of the various parts of the expression, does the curve approach this asymptote from above or below?

e) Find $\frac{dy}{dx}$. Find its value at $x = 0$, and thus confirm the result of the last question in the last part. Show that any turning point would have to satisfy $e^{1-x} = 1 - x$. By considering separately positive and negative values of u, show that the equation $e^u = u$ cannot have any real solutions, and thus that y does not have any turning points.

f) Sketch the curve.

Y5. Consider $y = x^2 e^{-x^2}$.

a) Note the symmetry of the function. Find any intercepts. Note what sign y must have. Find the limit as x goes to infinity.

b) Find $\frac{dy}{dx}$. Determine where it is zero, and hence find any turning points. Work out which are maxima and minima, first by considering the constraints on the shape of the curve, and then by examining the second derivative.

c) Sketch the curve.

Y6. Consider $y = x \sin(\frac{1}{x})$.

a) Show that y is constrained to lie between x and $-x$. Sketch these limiting lines on a graph.

b) Show that y is an even function of x.

c) Show that, as x tends to infinity, y tends to 1. *[Hint: consider the Taylor expansion of the sine for small values of its argument.]* Does it approach this asymptote from above or below? Mark the asymptote on your diagram.

d) Find the zeros (x-intercepts) of the function in terms of π. Mark the outermost few zeros on your graph. Determine the sign of the function between pairs of these zeros.

e) Hence sketch the graph. For the very inner part you may need to resort to scribbling.