IA NST Post-Christmas College Maths Test

January 2014

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a fresh sheet, writing your name and the question number clearly at the top. Calculators are forbidden.

Question 1

a) Find all of the complex roots of the equation $z^6 + 64 = 0$. Mark on an Argand diagram the points represented by these values of z. Draw on the circle on which all of the points lie. Identify which roots satisfy $z^3 = 8i$, and draw lines connecting their points together. Identify also any roots satisfying $z^3 = -8i$, and draw lines connecting their points together. [6]

b) A flat model star (perhaps made to the pattern of part (a)) of area A_0 is suspended from a ceiling, and spins back and forth about a vertical axis. It is illuminated by a horizontal beam of parallel light, casting a shadow on a wall behind it. The wall is normal to the light beam. The area of the shadow is given by

$$A = \left| \mathrm{Re} \left(e^{-3e^{-i\omega t}} A_0 e^{3e^{i\omega t}} \right) \right|$$

where t is time, and ω is a constant equal to $(\pi/6)\,\mathrm{s}^{-1}$.

i) Give an explicitly real formula for A in terms of t. [5]

ii) Sketch a graph of A against t for t between 0 and 12s. There is no need to find the exact positions of zeros and extrema except where these coincide with integer numbers of seconds. [7]

iii) Indicate on your sketch where the star is momentarily stationary, and where it is spinning fastest. [2]

Question 2

A Christmas tree is held in a pot full of rocks on a stand, inside a building. The origin of co-ordinates is at the centre of the base of the pot. The z-axis points vertically upwards. The stand has height 2, so the floor has equation $z + 2 = 0$. Unfortunately the rocks in the pot do not support the tree in the right way, so it is not vertical.

a) It is proposed to support the tree by means of a rope, tied to the tree at the point with position vector $\mathbf{s} = 7\mathbf{i}+24\mathbf{k}$, and suspended from the ceiling at a point with position vector $\mathbf{c} = 4\mathbf{i} + 1\mathbf{j} + 32\mathbf{k}$. Find a vector equation of the line followed by the rope. [2]

b) There is another rope in the building with the equation $\mathbf{r} = 29\mathbf{k} + \mu(\mathbf{i}+\mathbf{j})$; it is proposed to tie the two ropes together with a piece of string at their point of closest approach in order to provide more stability. Calculate the length of string needed. [7]

c) The curved surface of the green part of the tree can be described as the set of points with position vectors $\mathbf{r}$, where

$$-\frac{12}{13}|\mathbf{r} - \mathbf{s}| = \hat{\mathbf{s}} \cdot (\mathbf{r} - \mathbf{s})$$

with $\hat{\mathbf{s}}.(\mathbf{s}-\mathbf{r}) < 24$. As usual, $\hat{\mathbf{s}}$ is a unit vector in the direction of $\mathbf{s}$. Describe this surface as fully as possible, and calculate the distance of closest approach of these branches to the floor. (Recall that the floor has $z + 2 = 0$.) Give the distance as an exact decimal. [11]

Question 3

a) State Bayes' Theorem. [2]

b) In a certain town there are n homeless people in total sleeping on the streets on a particular night. They are distributed randomly throughout the town. An official from the Town Council counts carefully the number of homeless people sleeping on the streets, q, in precisely one quarter of the town. Find in terms of n the probability that q is (i) zero, (ii) one, and (iii) two. [3]

c) The official in fact finds $q = 1$. The Town Council do not know the value of n, and start by making the assumption that it is equally likely to be any number. (This is known as a *uniform prior*.) The probability worked out in (bii) is $P((q = 1)|\text{total } n)$. Use Bayes' Theorem to show that

$$P(\text{total } n|(q = 1)) = kn\left(\frac{3}{4}\right)^n$$

where k is an unknown constant. Since we know that $q = 1$, this is now the probability distribution for the unknown n. [3]

d) Find the value of k. In order to sum a series, you may find it helpful to consider the binomial expansion of $(1-x)^{-2}$. [7]

e) The Town Council do not understand Bayes' Theorem. They claim that n is most likely to be $4 \times 1 = 4$, so that the probability that $n > 4$ is less than 50%. (Legislation requires them to do something about the problem if it is likely that $n > 4$.) Find the actual probability that $n > 4$, and comment. [5]

Question 4*

Define $S_n = \sum_{r=1}^{n} \frac{1}{r}$, the sum of the reciprocals of the first n natural numbers.

a) Show that $\lim_{n\to\infty} S_n$ is undefined. [3]

b) The *floor function*, $\lfloor x \rfloor$, returns the largest integer which is less than or equal to x. Thus, for example, $\lfloor 7 \rfloor = 7$, $\lfloor 7.2 \rfloor = 7$, and $\lfloor 7.8 \rfloor = 7$. Sketch $y = \frac{1}{x}$ and $y = \frac{1}{\lfloor x \rfloor}$ on the same axes between $x = 1$ and $x = 7$. [4]

c) By considering the graphs sketched above, show that the limit defining

$$\gamma = \lim_{n\to\infty} (S_n - \log_e n)$$

exists and show further that $0 < \gamma < 1$. This mathematical constant γ is known as the Euler-Mascheroni constant. [7]

d) In a dice game, an icosahedral (twenty-sided) die is thrown repeatedly. In the original game, the player must throw the die repeatedly until he or she throws a 1, then again until a 2 is thrown, and so on until all twenty of the numbers have appeared in the correct order (whichever other numbers intervene). A new version of the game is suggested in which all twenty numbers must be thrown, but the order is unimportant: the game is over once all twenty numbers have appeared at some point. Using $\gamma = 0.58$ and $\log_e 20 = 3.00$ (both correct to two decimal places) find an approximation to how much shorter the expected length of the new version of the game is compared to the expected length of the original version. Express your answer as a percentage of the expected length of the original game, correct to the nearest whole percent. [6]

[When the probability of success on a given trial is p, the number of trials needed to achieve the first success follows a distribution known as the geometric distribution. You may use the fact that the mean of the geometric distribution is $1/p$.]