IA NST Post-Christmas College Maths Test

January 2021

You have one hour. Complete any **two** questions from those printed below. The starred question requires knowledge of B-course material. The number of marks available for each part of a question is shown in square brackets. Begin your answer to each question on a new page, writing the question number clearly at the top. Do not submit answers to more than two questions. Calculators may not be used for this test.

Question 1

a) Find the equation of the plane Π through the points $(7, 3, 7)$, $(1, 1, 0)$, and $(1, 5, 2)$ in the form $px+qy+rz = s$ where (x, y, z) is the general position vector of a point on the plane. [4]

b) Point A is at the point $(7, 3, 4)$. Find the distance of point A from the plane Π. [4]

c) One end of a piece of stretched elastic is tied to point A. The other end is tied to point B which is at $(7, 5, 2)$. Find the length of the shadow cast by the elastic on plane Π if it is illuminated by a beam of light perpendicular to plane Π. [5]

d) One end of a piece of string is tied to the point A, and the other end is tied to a point on plane Π, chosen such that the elastic and the string form part of a straight line. Find the angle that the string makes with plane Π. [2]

e) Point B moves directly towards plane Π by some small distance ϵ, whilst point A remains in the same place. The elastic adjusts slightly in length but remains under tension. The string does not move. Find an approximation to the angle between the elastic and the string, to first order in the small quantity ϵ. [5]

Question 2

The displacement x of a particle from its equilibrium position is given by

$$x = \mathrm{Re}(x_0 e^{i\omega t})$$

where t is time, ω a positive real constant, and x_0 a complex constant.

a) The maximum value of x is 10m and the value of x when $t = 0$ is -8m. When $t = 0$, x is increasing. Find x_0 in the form $p + iq$. [4]

b) Given that x obeys the equation

$$m\frac{\mathrm{d}^2 x}{\mathrm{d}t^2} + kx = 0$$

where $m = 2$kg and $k = 18\mathrm{kg\,s^{-2}}$, find ω. [3]

c) What is the minimum value of x? [1]

d) What time Δt_x elapses between the first and second occasions when $x = 0$? [2]

The displacement y of another particle from its equilibrium position is given by

$$y = \mathrm{Re}(y_0 e^{i\eta t})$$

where t is time, $y_0 = x_0$, and η is a complex constant with a positive real part.

e) Given that y obeys the equation

$$m\frac{\mathrm{d}^2 y}{\mathrm{d}t^2} + b\frac{\mathrm{d}y}{\mathrm{d}t} + ky = 0$$

where $4mk > b^2$, find an exact formula for η in terms of m, b, and k. [3]

f) Using the values of m and k from part (b) and $b = 0.12\mathrm{kg\,s^{-1}}$, find (in exact surd form) Δt_y, the time elapsed between the first and second occasions when $y = 0$. [4]

g) By using an appropriate first-order expansion, find the approximate fractional difference of Δt_y from Δt_x. [3]

Question 3

A certain pandemic virus V affects students randomly, never causing symptoms, at a known rate which is different at different colleges. Ten students from Lazarus College are to visit the Museum of Probability as part of their course. The probability that a student from Lazarus College carries virus V is only 0.01. However, the students must still all be tested for virus V before entering the Museum, lest the visit become a "super-spreader" event. Those that test positive will not enter. (Thus, although ten students take the test, it may be that fewer than ten enter the Museum.) The Director of the Museum buys cheap tests from Ruritania. These tests give the correct result (positive for those with virus V, negative for those without) with probability 0.9, independently of whether the person tested has the virus or not, and independently of whether they have given the correct result on other occasions.

a) Oscar is a Lazarus student who tests positive, and is therefore denied entry to the Museum. Find the probability that he carries virus V. [4]

b) Find the expected number of students from Lazarus College who enter the Museum. [3]

Ten students from Shrewsbury College, at the same university, are also to visit the Museum at the same time. The probability that a student at Shrewsbury College carries virus V is 0.1, which is very high. Therefore the Director of the Museum decides that students from Shrewsbury College must take the test twice, and that only those who test negative twice may enter the Museum.

c) Lucinda is a Shrewsbury student who tests positive at least once, and is therefore denied entry to the Museum. Find the probability that she carries virus V. [4]

d) One of the students who have entered the Museum is randomly selected to dress up in a goat costume as part of an enactment of the Monty Hall problem. (The Museum possesses one actual goat of its own, but the Monty Hall problem requires two goats.) Find the probability that the student in the costume is from Lazarus College. [4]

e) The event is declared to be a Disaster if at least one of the students admitted does carry virus V. The Director of the Museum claims that the measures he has taken mean that the probability of a Disaster is precisely 0.02. By using an appropriate expansion, show that the Director is approximately correct, and, by considering the next term in the expansion, give an approximation to the error in his estimate of the probability of a Disaster. [5]

Question 4*

a) Prove that

$$\left(\int_0^1 f(x)g(x)\,\mathrm{d}x\right)^2 \leq \int_0^1 (f(x))^2\,\mathrm{d}x \int_0^1 (g(x))^2\,\mathrm{d}x$$

for any functions f and g. [8]

b) Show that

$$\frac{16}{\pi^3} \leq \int_0^1 \cos\left(\frac{\pi x}{2}\right)(1-x^2)^{\frac{3}{4}}\,\mathrm{d}x \leq \sqrt{\frac{3\pi}{32}} \quad . \quad [12]$$